$SO(n+1)$ SYMMETRIC SOLUTIONS OF THE EINSTEIN EQUATIONS IN HIGHER DIMENSIONS

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Abstract

A method of solving the Einstein equations with a scalar field is presented. It is applied to find higher dimensional vacuum metrics invariant under the group $SO(n+1)$ acting on n-dimensional spheres.

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1 Introduction

Recently, higher dimensional solutions of the Einstein equations became important because of a great interest in string theories and induced effective theories in 4+d dimensions (see e.g. [1] and references therein). In the brane-world gravity matter fields are usually confined to a 4-dimensional brane and gravity can propagate in extra dimensions. At the classical level the gravitational field of a bulk should satisfy the vacuum Einstein equations, possibly with a cosmological constant. In order to find and classify higher dimensional solutions methods of the standard general relativity were generalized (see [2, 3, 4] and references therein).

One of the most effective techniques of solving the Einstein equations is their reduction via symmetries. Taking into account a role of the Birkhoff theorem in general relativity it is not surprising that much attention is payed to higher dimensional metrics admitting symmetries of 2-dimensional or higher dimensional sphere. For instance, this property is shared by the 5-dimensional Gross-Perry metric [5] studied in the framework of the Kaluza-Klein theory (see [6] for a recent discussion of this metric). Recently, a more systematic investigation of 5-dimensional $SO(3)$ symmetric vacuum metrics was performed by Lake [7] and Millward [8].

In this paper we consider $(n+N+1)$- dimensional metrics invariant under the rotation group $SO(n+1)$ acting on n-dimensional spheres. Following the Kaluza-Klein approach (see e.g. [9]) we first recall the dimensional reduction of the Einstein equations to equations in $N+1$ dimensions with a scalar field ϕ and an exponential potential. In section 3 we reduce the latter equations, with a general potential, under the assumption that surfaces $\phi = const$ are Einstein spaces and their normal vector field is geodetic. We obtain a closed system of two ordinary differential equations (they correspond to the Friedmann equations in cosmology) and algebro-geometric conditions on the metric of the surfaces. For $N = n = 2$ they describe a class of metrics which generalizes those

considered in [7, 8]. Finally, we present examples of vacuum metrics obtained by our method. Among them there are new solutions which belong to the generalized Kundt class [10].

2 Symmetry reduction of the Einstein equations with a cosmological constant

Let M be a $(n+N+1)$–dimensional manifold with a Lorentzian metric g admitting $SO(n+1)$ spherical symmetry. We assume that orbits of the group are diffeomorphic to the n-dimensional sphere S_n. In local coordinates $x^\mu = \left\{x^a, x^A\right\}$, $a = 0, 1, \cdots, N$; $A = N+1, \cdots, N+n$, the metric can be written in the following form

$$g = g_{ab}dx^a dx^b - e^{2f} s_{AB} dx^A dx^B \ , \tag{1}$$

where $s_{AB}dx^A dx^B$ is the standard metric of S_n and g_{ab} and f are functions of coordinates x^a (note that for n=1 (1) is not the most general invariant metric).

Components of the Ricci tensor of (1) read

$$\begin{aligned} R_{aB} &= 0 & (2)\\ R_{AB} &= \left(n - 1 + \Box' f + n f^{|a} f_{|a}\right) s_{AB} & (3)\\ R_{ab} &= R'_{ab} - n f_{|a} f_{|b} - n f_{|ab} \ . & (4) \end{aligned}$$

Here $_{|a}$, $R^{'}_{ab}$ and $\Box'$ denote, respectively, the covariant derivative, the Ricci tensor and the d'Alembert operator of metric $g' = g_{ab}dx^a dx^b$.

The vacuum Einstein equations with a cosmological constant Λ in $(n+N+1)$ dimensions are equivalent to

$$R_{\mu\nu} = \frac{2\Lambda}{1-N-n} g_{\mu\nu} \ . \tag{5}$$

For $N = 1$ solutions of (5) of the form (1) are multidimensional Schwarzschild-de Sitter metrics

$$g = \left(1 - \frac{2M}{r^{n-1}} - \frac{2\Lambda r^2}{n\,(n+1)}\right) dt^2 - \left(1 - \frac{2M}{r^{n-1}} - \frac{2\Lambda r^2}{n\,(n+1)}\right)^{-1} dr^2 - r^2 s_{AB} dx^A dx^B. \tag{6}$$

In what follows we assume $N > 1$. In this case we can apply to $g^{'}$ a conformal transformation of the form

$$\widetilde{g}_{ab} = e^{\frac{2n}{N-1} f} g_{ab} \tag{7}$$

which induces the following changes

$$\widetilde{R}_{ab} = R'_{ab} - n f_{|ab} + \frac{n^2}{N-1} f_{|a} f_{|b} - \frac{n}{N-1}(\Box' f + n f^{|c} f_{|c}) g_{ab} \tag{8}$$

$$\widetilde{\Box} f = e^{-\frac{2n}{N-1} f}(\Box' f + n f^{|c} f_{|c}) \ . \tag{9}$$

Here $\widetilde{\Box} f=\widetilde{g}^{ab}f_{;ab}$ and the semicolon denotes the covariant derivative with respect to metric $\widetilde{g}$.

By virtue of (8) and (9) expresions (3), (4) take the form

$$\begin{aligned}R_{AB} &= \left(n-1+e^{2(\frac{n+N-1}{N-1}f)}\widetilde{\Box} f\right) s_{AB} &\quad (10)\\ R_{ab} &= \widetilde{R}_{ab}-n\frac{n+N-1}{N-1}f_{;a}f_{;b}+\frac{n}{N-1}\widetilde{\Box} f\widetilde{g}_{ab} &\quad (11)\end{aligned}$$

and equations (5) reduce to

$$\widetilde{\Box} f=-\left(n-1\right)e^{-2\frac{n+N-1}{N-1}f}+\frac{2\Lambda}{n+N-1}e^{-\frac{2n}{N-1}f} \tag{12}$$

and

$$\widetilde{R}_{ab}=n\frac{n+N-1}{N-1}f_{;a}f_{;b}-\left(\frac{n}{N-1}\widetilde{\Box} f+\frac{2\Lambda}{n+N-1}e^{-\frac{2n}{N-1}f}\right)\widetilde{g}_{ab}\ . \tag{13}$$

We substitute (12) into (13) and we pass to the Einstein tensor $\widetilde{G}_{ab}$ of $\widetilde{g}$. In this way we obtain

$$\widetilde{G}_{ab}=n\frac{n+N-1}{N-1}\left(f_{;a}f_{;b}-\frac{1}{2}f^{;c}f_{;c}\widetilde{g}_{ab}\right)+\left(-\frac{1}{2}n\left(n-1\right)e^{-2\frac{n+N-1}{N-1}f}+\Lambda e^{-\frac{2n}{N-1}f}\right)\widetilde{g}_{ab}. \tag{14}$$

After rescaling

$$\phi=\sqrt{\frac{n\left(n+N-1\right)}{\left(N-1\right)}}f \tag{15}$$

equations (12) and (14) take the form of $(N+1)$–dimensional Einstein equations with the scalar field ϕ

$$\widetilde{G}_{ab}=\phi_{;a}\phi_{;b}+\left(V(\phi)-\frac{1}{2}\phi^{;c}\phi_{;c}\right)\widetilde{g}_{ab} \tag{16}$$

$$\widetilde{\Box}\phi=-V_{,\phi}\ . \tag{17}$$

The potential V is given by

$$V=-\frac{1}{2}n\left(n-1\right)e^{-2\sqrt{\frac{n+N-1}{n(N-1)}}\phi}+\Lambda e^{-2\sqrt{\frac{n}{(N-1)(n+N-1)}}\phi}\ . \tag{18}$$

For a non constant function ϕ equation (17) follows from (16) and it is equivalent to the energy-momentum conservation law $T^{ab}{}_{;b}=0$.

3 Reduction of the Einstein equations with a scalar field

Let us consider equations (16), (17) in spacetime of dimension $N+1\geq 3$. Assume that $\phi_{,a}\neq 0$ and surfaces ϕ=const are not null. Then there are coordinates ϕ, x^i such that

$$\tilde{g}=\tilde{g}_{\phi\phi}d\phi^2+\tilde{g}_{ij}dx^idx^j\ . \tag{19}$$

The coordinate ϕ is timelike if $\tilde{g}_{\phi\phi} > 0$. In this case we set $x^0 = \phi$ and $i = 1, ...N$. If $\tilde{g}_{\phi\phi} < 0$ ϕ is spacelike, $x^N = \phi$ and $i = 0, .., N-1$.

Assume moreover that $\tilde{g}_{\phi\phi}$ is independent of x^i. Then we can find a new coordinate s such that

$$\phi = \phi(s) \tag{20}$$

and

$$\tilde{g} = \epsilon ds^2 + \tilde{g}_{ij} dx^i dx^j \ , \ \epsilon = \pm 1 \ . \tag{21}$$

Geometrically, above assumptions mean that the field of normal vectors to surfaces ϕ=const is geodesic, timelike or spacelike, and s is the affine parameter along the field.

Under conditions (20), (21) equation (17) yields

$$\ddot{\phi} + \dot{\phi}(\ln \sigma)^{\cdot} = -\epsilon V_{,\phi} \ , \tag{22}$$

where

$$\sigma = |det(\tilde{g}_{ij})|^{\frac{1}{2}} \tag{23}$$

and the dot denotes the partial derivative with respect to s. It follows from (22) that

$$\sigma = \beta(s)\sigma_0(x^i) \tag{24}$$

and

$$(\beta\dot{\phi})^{\cdot} = -\epsilon\beta V_{,\phi} \ , \tag{25}$$

where β is a function independent of coordinates x^i and σ_0 is independent of s.

The Einstein tensor of metric (21) takes the following form

$$\tilde{G}^{\phi}{}_{\phi} = -\frac{1}{2}\hat{R} + \epsilon\sigma^{-2}\Pi \tag{26}$$

$$\tilde{G}^{\phi}{}_{j} = \epsilon(\sigma^{-1}\pi^{k}{}_{j})_{;k} \tag{27}$$

$$\tilde{G}^{i}{}_{j} = \hat{G}^{i}{}_{j} - \epsilon\sigma^{-1}\dot{\pi}^{i}{}_{j} - \epsilon\sigma^{-2}\Pi\delta^{i}{}_{j} \tag{28}$$

where

$$\Pi = \frac{1}{2}\Big[\frac{(\pi^{i}{}_{i})^2}{N-1} - \pi^{i}{}_{j}\pi^{j}{}_{i}\Big] \ , \tag{29}$$

quantities

$$\pi^{i}{}_{j} = \frac{1}{2}\sigma\tilde{g}^{ik}\dot{\tilde{g}}_{kj} - \dot{\sigma}\delta^{i}{}_{j} \tag{30}$$

are related to the exterior curvature of surfaces s=const and $\hat{G}^{i}{}_{j}$ and $\hat{R}$ are, respectively, the Einstein tensor and the Ricci scalar of the metric

$$\hat{g} = \tilde{g}_{ij} dx^i dx^j. \tag{31}$$

In order to simplify the r.h.s. of (28) let us assume that

$$\hat{G}^{i}{}_{j} = \hat{\Lambda}\delta^{i}{}_{j} \tag{32}$$

(note that $\hat{\Lambda} = 0$ if $N = 2$). It follows from (32) and the Bianchi identities that $\hat{\Lambda} = \hat{\Lambda}(s)$. Equations (16) with indices i, j now yield

$$\dot{\pi}^i{}_j = \sigma(\epsilon\hat{\Lambda} - \sigma^{-2}\Pi + \frac{1}{2}\dot{\phi}^2 - \epsilon V)\delta^i{}_j \ . \tag{33}$$

Since $\dot{\pi}^i{}_j \sim \delta^i{}_j$ the matrix $\pi^i{}_j$ must have the following structure

$$\pi^i{}_j = a\delta^i{}_j + P'^i_j(x^k) \ , \quad P'^i{}_i = 0. \tag{34}$$

The function a can be related to β and σ_0 by substituting (34) and (24) into the identity

$$\pi^i{}_i = (1 - N)\dot{\sigma} \tag{35}$$

which follows from (30). Consecutively we obtain

$$\pi^i{}_j = \sigma_0[(\frac{1}{N} - 1)\dot{\beta}\delta^i{}_j + P^i{}_j(x^k)] \ , \tag{36}$$

where

$$P^i{}_i = 0 \tag{37}$$

and the matrix $P = (P^i{}_j)$ is independent of s. Substituting (36) back to (33) yields a condition on P

$$P^i{}_j P^j{}_i = 2c = const \tag{38}$$

and the following equation for the functions $\beta(s)$ and $\phi(s)$

$$(\frac{1}{N} - 1)(\frac{\ddot{\beta}}{\beta} - \frac{\dot{\beta}^2}{2\beta^2}) - \frac{c}{\beta^2} - \frac{1}{2}\dot{\phi}^2 + \epsilon(V - \hat{\Lambda}) = 0 \ . \tag{39}$$

Given (36) and (24) relation (30) becomes a linear equation for the matrix $\hat{g} = (\tilde{g}_{ij})$ (if there is no confusion we denote a metric and the corresponding matrix of its components by the same symbol). Its general solution has the form

$$\hat{g} = \beta^{2/N}\gamma(x^i)e^{P\tau(s)} \ , \tag{40}$$

where $\gamma = (\gamma_{ij})$ is a nondegenerate matrix independent of s and function τ is related to β via

$$\beta\dot{\tau} = 2. \tag{41}$$

In order to guarantee that the r.h.s. of (40) is a symmetric matrix we have to require

$$\gamma_{ij} = \gamma_{ji} \ , P_{ij} = P_{ji} \ , \tag{42}$$

where $P_{ij} = \gamma_{ik}P^k{}_j$. Note that equation (40) implies (24) with $\sigma_0 = |det\gamma|^{1/2}$.

Thus, under assumptions (20), (21) and (32) equation (17) takes the form (25) and equations (16) with indices i, j are equivalent to (37)-(42).

Let us consider now equations (16) with indices a, ϕ. From the point of view of an evolution with respect to the coordinate s these equations are constraints. For indices i, ϕ they take the form

$$P^k_{\ i;k} = 0 \ . \tag{43}$$

Due to (37) and (38) the s derivative of the l.h.s. of (43) vanishes. Hence, it is sufficient to solve (43) with covariant derivatives defined by the s-independent metric γ.

By virtue of (26), (29), (36)-(38) equation (16) with indices ϕ, ϕ is equivalent to

$$\epsilon \hat{R} = (1 - \frac{1}{N})\frac{\dot{\beta}^2}{\beta^2} - \frac{2c}{\beta^2} - \dot{\phi}^2 - 2\epsilon V \ . \tag{44}$$

It follows from (44) that $\hat{R}$ cannot depend on coordinates x^i. Taking the s derivative of (44) and comparing with (39) yields

$$\beta^{2/N}\hat{R} = const \ . \tag{45}$$

Equations (32) and (45) can be jointly written as the following condition on the Ricci tensor of the metric $\gamma e^{P\tau}$

$$R^i_{\ j}(\gamma e^{P\tau}) = \lambda \delta^i_{\ j} \ , \quad \lambda = const \ . \tag{46}$$

Due to (46) equation (44) is the first integral of (39) and equation (39) can be postponed if $\dot{\beta} \neq 0$. If β=const equations (25), (39), (41) and (44) admit solutions only if V=const. In this case, without a loss of generality, we can assume that

$$\beta = 1 \ , \quad V = -\frac{1}{2}(N-1)\lambda \tag{47}$$

$$\tau = 2s \ , \quad \phi = s\sqrt{-2c - \epsilon\lambda} \ , \quad 2c < -\epsilon\lambda \ . \tag{48}$$

Summarizing, in order to construct a class of solutions of the Einstein equations with a scalar field and a nonconstant potential V we can proceed along the following steps:

- Find N-dimensional metric γ_{ij} and a symmetric tensor P_{ij} such that conditions (46), (37), (38) and (43) (with covariant derivatives defined by γ) are satisfied.
- Find solutions ϕ, $\beta \neq$ const of ordinary differential equations (25), (44).
- Construct (N+1)-dimensional metric according to (21), (40) and (41).

It follows from the reduction in section 2 that for V given by (18) metric $\tilde{g}$ and the scalar field ϕ define a $(n+N+1)$-dimensional Einstein metric. In this case g_{ab} and f are given by (7) and (15).

4 Examples

For any dimension $N > 1$ the conditions on γ and P from section 3 are obviously fulfilled by

$$\gamma_{ij} = \gamma_{ji} = const \ , \quad P_{ij} = P_{ji} = const \ , \quad P^i{}_i = 0 \ . \tag{49}$$

For $\epsilon = 1$ (49) yields the Misner parametrization [13] of the Bianchi I cosmological models. In this case we can assume that $\gamma_{ij} = -\delta_{ij}$ and P is diagonal. For $\epsilon = -1$ we can transform γ into the N-dimensional Minkowski metric. In this case the matrix P can be simplified by means of N-dimensional Lorentz transformations. For N=2 one obtains the following canonical forms of the metric $\gamma e^{P\tau}$

$$e^{a\tau} dt^2 - e^{-a\tau} dx^2 \tag{50}$$

$$\cos{(a\tau)}(dt^2 - dx^2) + 2\sin{(a\tau)} dt dx \tag{51}$$

$$du(dv + a\tau du). \tag{52}$$

For N=3 they read

$$e^{a\tau} dt^2 - e^{b\tau} dx^2 - e^{-(a+b)\tau} dx^2 \tag{53}$$

$$e^{b\tau} \cos{(a\tau)}(dt^2 - dx^2) + 2e^{b\tau} \sin{(a\tau)} dt dx - e^{-2b\tau} dy^2 \tag{54}$$

$$du(e^{b\tau} dv + a\tau du) - e^{-2b\tau} dy^2. \tag{55}$$

Here $t,\ x$, or $u,\ v$, and y are coordinates and $a,\ b$ are constants. To simplify P and $\gamma e^{P\tau}$ for $N > 3$ a classification of symmetric tensors in Lorentz manifolds can be usefull (see [11, 12] and references therein). Metrics considered in [5, 7, 8] are related to particular realizations of (50).

In the case $N = 2$ we can find general solution conditions for γ and P. Indeed, for $N = 2$ equation (32) is identically satisfied with $\hat{\Lambda} = 0$. For any N the s derivative of the l.h.s. of (45) vanishes by virtue of (37) and (43). Hence, for $N = 2$ condition (46) is equivalent to the requirement that metric γ_{ij} has a constant curvature. For instance, if $\epsilon = -1$, γ reads

$$\gamma = \frac{du dv}{(1 + \frac{\lambda}{4} uv)^2} \ , \tag{56}$$

where $u,\ v$ are null coordinates and λ is a constant. Given (56) conditions (37), (38) and (43) can be fully solved. If $c \neq 0$ then $\lambda = 0$ and $\gamma e^{P\tau}$ is given by (50) or (51).

If $c = 0$ one obtains

$$\gamma e^{P\tau} = \frac{du dv}{(1 + \frac{\lambda}{4} uv)^2} + \tau h(u) du^2 \ , \tag{57}$$

where h is an arbitrary function of u. Note that (57) leads to vacuum metrics (1), which belong to the generalized Kundt class [10].

Let us consider equations (25) and (44) with $\epsilon = -1$ and the potential V given by (18) with $\Lambda = 0$. For $n = 1$ $V = 0$ and these equations can be solved up to quadratures since equations (25) and (41) imply

$$\dot{\phi} = \frac{2c'}{\beta} \ , \ c' = const \tag{58}$$

$$\phi = c'\tau \tag{59}$$

and (44) yields

$$s = \pm \int \frac{\sqrt{1 - \frac{1}{N}} d\beta}{\sqrt{N\lambda\beta^{2\left(1-\frac{1}{N}\right)} + 2c + c'^2}} \ . \tag{60}$$

In the case (49) $\lambda = 0$ and $\beta \sim s$. From (58)-(60) one obtains vacuum metrics (1) with components depending on one spacelike coordinate s (the same is true if $\beta = const$, see (47) and (48)). They are not particularly interesting from point of our method since they can be obtained by a straightforward integration of the Einstein equations. For $N = 2$ they belong to the class of metrics found by Kasner [14].

In the case (57) for $N = 2$ equations (58)-(60) can be solved analitycally. For $\lambda \neq 0$ one is led to the following 4-dimensional vacuum solution of the Einstein equations

$$g = \epsilon' \left(s + s_0\right)^2 du \left(\frac{\lambda dv}{(1 + \frac{\lambda}{4}uv)^2} + \ln \left| \frac{s - s_0}{s + s_0} \right| h'(u) du \right)$$
$$- \left| \frac{s + s_0}{s - s_0} \right| ds^2 - \left| \frac{s - s_0}{s + s_0} \right| d\varphi^2 \ , \tag{61}$$

where $\epsilon' = \pm 1$ is the sign of $(s^2 - s_0^2)$, $s_0 = \frac{c'}{\sqrt{2\lambda}}$ is a constant and $h' = h/s_0$ is an arbitrary function of u. The constant λ can be replaced by any nonzero value without a loss of generality. The vector field ∂_v generates a null geodesic shear-free congruence with no twist and expansion. Metric (61) belongs to a class of the Kundt metrics found by Kramer and Neugebauer [15]. This class contains also the following metrics obtained from (57)-(60) for $c = 0$ and $\lambda = 0$

$$g \ = \ \epsilon' du \left(dv + \ln|s| h(u) du\right) - |s|^{-1} ds^2 - |s| \, d\phi^2 \ , \tag{62}$$

where now $\epsilon' = \pm 1$ is the sign of s.

If $\epsilon = -1$, $\Lambda = 0$ and $n > 1$ simple solutions of (25) and (44) can be obtained under the assumption that V and β have the form as^b, where a and b are constants. In this case one obtains $c = 0$ and

$$e^{-2\sqrt{\frac{n+N-1}{n(N-1)}}\phi} = \frac{(N-1)^2}{(n+N-1)^2} s^{-2} \ , \quad \beta = \beta_0 s^{\frac{nN}{n+N-1}} \ , \quad \lambda = 0 \tag{63}$$

or

$$e^{-2\sqrt{\frac{n+N-1}{n(N-1)}}\phi} = \frac{(N-1)^2}{(n-1)(n+N-1)} s^{-2} \ , \ \ \beta = \beta_0 s^N \ , \ \ \lambda = -\frac{(N-1)^2}{n+N-1} \beta_0^{2/N} \ , \tag{64}$$

where $\beta_0 = const$. Let r be a new coordinate given by

$$r = s^{\frac{N-1}{n+N-1}}. \tag{65}$$

After a minor reparametrization of variables u, v and h, for $N = 2$ one obtains from (57), (64) and (63) the following vacuum metrics

$$g = du\left(dv - r^{1-n}h(u)du\right) - dr^2 - r^2 s_{AB}dx^A dx^B \tag{66}$$

$$g = du\left(\frac{4r^2 dv}{(n+1)(1-uv)^2} - r^{1-n}h(u)du\right) - dr^2 - \frac{n-1}{n+1}r^2 s_{AB}dx^A dx^B\ . \tag{67}$$

They are examples of $(n+3)$–dimensional generalized Kundt metrics [2, 10]. In both cases the vector ∂_v defines a shear- and twist-free congruence of null geodesics. Metric (66) has vanishing scalar invariants [16]. It is particular case of generalized pp wave of type N. As such it can be easily obtained by standard methods of general relativity [11]. It tends to the Minkowski metric when $r \to \infty$. Thus, it is asymptotically flat on any timelike section given by $u = u(t),\ v = v(t)$. For instance, if $n = 2$ the section $u = v = t$ is 4-dimensional and the corresponding Newton potential takes the form $h(t)/r$. An interpretation of this metric within the brane-world gravity is unclear since the exterior curvature of the section does not yield any reasonable energy-momentum tensor.

Metric (67) is of type II in generalized Petrov classification [2]. Its Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ nowhere vanishes and it is proportional to r^{-4}. Thus, this metric is an example, perhaps the only known explicitly, of a multidimensional Kundt metric with nonconstant scalar invariants (see [17] for a discussion of metrics with constant invariants). This metric is singular at $r = 0$ and at $uv = 1$. The latter singularity can be moved to infinity by means of the transformation

$$u' = u^{-1}\ , \quad v' = -\frac{2ur^2}{(n+1)(1-uv)} \tag{68}$$

which puts the metric into the following form

$$g = du'\left(2dv' - \frac{4v'}{r}dr - \left((n+1)\frac{v'^2}{r^2} + r^{1-n}h'(u')\right)du'\right) - dr^2 - \frac{n-1}{n+1}r^2 s_{AB}dx^A dx^B\ . \tag{69}$$

Solution (63) can be also merged with (49) for $N > 2$ provided $TrP^2 = 0$. In this way one obtains vacuum metrics of the form

$$g = (\gamma e^{Pr^{1-n}})_{ij}dx^i dx^j - dr^2 - r^2 s_{AB}dx^A dx^B\ . \tag{70}$$

For instance, for N=3, substituting (54) with $a = \pm\sqrt{3}b$ into (70) yields the following $(n+4)$-dimensional metric singular at $r = 0$

$$e^{br^{1-n}}\cos(\sqrt{3}br^{1-n})(dt^2 - dx^2) + 2e^{br^{1-n}}\sin(\sqrt{3}br^{1-n})dtdx - e^{-2br^{1-n}}dy^2 - dr^2 - r^2 s_{AB}dx^A dx^B. \tag{71}$$

5 Summary

We have considered multidimensional metrics (1) invariant under the group $SO(n+1)$ acting on n-dimensional spheres. For these metrics, we have reduced vacuum Einstein equations with cosmological constant to lower dimensional Einstein equations with a scalar field. In section 3 we proposed an ansatz which simplifies these equations for any potential of the scalar field. Our method is summarized at the end of section 3. Using this approach in section 4 we were able to rediscover known vacuum solutions of the form (1) and to find new ones (see e.g. (67) and (70)). Note that equations (25), (44) do not depend on details of the matrices γ and P except the trace of P^2. Thus, it might be possibile to generalize already known solutions if they satisfy assumptions of our method.

The presented reduction of the Einstein equations is different from that in brane-world gravity [18, 19]. In the framework of this theory our method can be used to find $SO(n+1)$ symmetric bulk metric on one side of a brane. This metric can be extended to the other side in such a way that the exterior curvature has a jump corresponding to matter fields located on the brane (see e.g. examples in [19]). It is highly nontrivial to obtain such a configuration which describes a physically realistic situation (work in progress).

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